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Discrete Mathematics 287 (2004) 165–170
www.elsevier.com/locate/disc

Note

Biclique comparability digraphs of bipartite graphs and minimum ranks of partial matrices

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Received 21 August 2002; accepted 15 July 2004

Abstract

Every bipartite graph has a *biclique comparability digraph* whose vertices are the inclusion-maximal complete bipartite subgraphs of the bipartite graph and whose arcs correspond to inclusions of the relevant color classes. I characterize those digraphs that correspond to bipartite graphs and, in particular, those that correspond to chordal bipartite graphs.

This is motivated by work on finding the minimum rank of completions of partially specified matrices. In particular, Woerdeman (Integral Equations Operator Theory 10 (1987) 859) proved a formula for minimum rank in special cases that can be naturally reformulated in terms of the biclique comparability digraphs of the bipartite graphs that have the partial matrices as incidence matrices. Cohen et al. (Oper. Theory Adv. Appl. 40 (1989) 165) conjecture that this formula actually gives the minimum rank if and only if the corresponding bipartite graph is chordal bipartite.

Keywords: Comparability graphs; Bipartite graphs; Chordal bipartite graphs; Minimum ranks; Partial matrices

0. Introduction

Section 1 develops the concepts of biclique comparability (di)graphs of bipartite graphs and characterizes those (di)graphs that correspond to (chordal) bipartite graphs. Section 2 then translates the motivating work on ranks of partial matrices into this graph-theoretic language, including Woerdeman's (1987) formula for triangular matrices. (The new formulation resembles basic formulas from chordal graph theory as developed in [7].) Section 2 closes with the 1989 conjecture that this formula actually characterizes when the associated bipartite graph is chordal bipartite.

1. Biclique comparability graphs

Suppose G is a bipartite graph whose n vertices are grouped into 'row' and 'column' color classes (anticipating the matrix applications in Section 2). Let $\mathscr{B} = \mathscr{B}(G)$ be the set of all inclusion-maximal complete bipartite subgraphs ($K_{a,b}$s) of G.

Form the *biclique comparability digraph* $\vec{H} = \vec{H}(G)$ having vertex set $\mathscr{B}$ with an arc from G_i to G_j if and only if all the column vertices of G_i are contained in G_j. Because of the maximality of members of $\mathscr{B}$, having an arc from G_i to G_j is also equivalent to having all the row vertices of G_j contained in G_i and, whether using column vertices or row vertices, these containments will always be proper.

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doi:10.1016/j.disc.2004.07.004

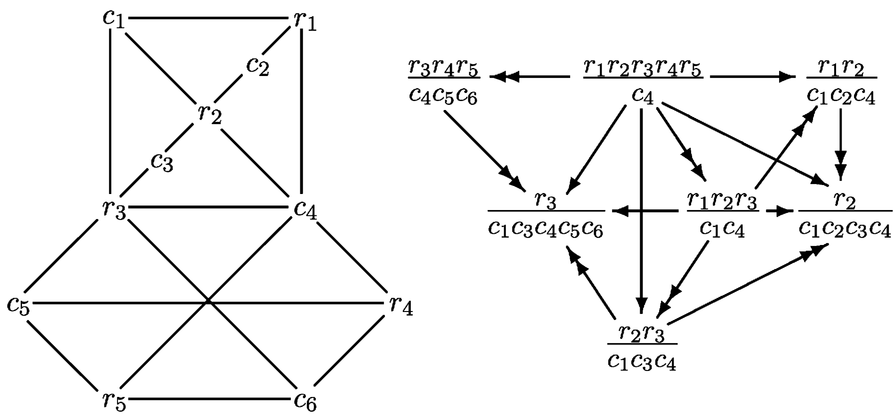

Fig. 1. A bipartite graph G and with its biclique comparability digraph $\vec{H}(G)$; the double-headed arcs form the transitive reduction of $\vec{H}(G)$.

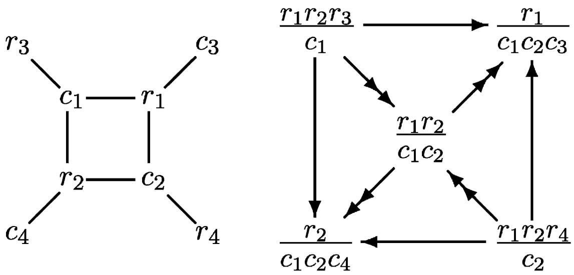

Fig. 2. Another bipartite graph G with its biclique comparability digraph $\vec{H}(G)$; the double-headed arcs form the transitive reduction of $\vec{H}(G)$.

For instance, Fig. 1 shows a bipartite graph G with its biclique comparability digraph $\vec{H}(G)$. In it, $\frac{r_1r_2}{c_1c_2c_4}$, for example, denotes the complete bipartite $K_{2,3}$ subgraph with two row vertices r_1, r_2, three column vertices c_1, c_2, c_4, and six edges r_ic_j ($i \in \{1, 2\}$, $j \in \{1, 2, 4\}$).

Because the set of row [respectively, column] vertices of the complete bipartite graphs corresponding to the ends of an arc of $\vec{H}(G)$ always exhibit proper containment, biclique comparability digraphs will be *asymmetric*: there cannot be arcs going both ways between two vertices. The *biclique comparability graph* $H(G)$ is the undirected simple underlying graph of $\vec{H}(G)$. Comparability graphs are a well-studied class of graphs [7, Section7.6], often called *containment graphs* or *transitively orientable graphs*; $\vec{H}(G)$ is the transitive orientation of $H(G)$.

An *anticycle* of a directed graph is a cycle that consists of an even number of arcs with consecutive arcs oppositely directed; in other words, every vertex has either in-degree 2 or 0 within the cycle. By transitivity, every induced cycle in $\vec{H}(G)$ of length greater than three must be an anticycle. Fig. 2 shows an example with a length-4 anticycle that is the 'rim' of a 4-spoked wheel with center $\frac{r_1r_2}{c_1c_2}$. As another example, $\vec{H}(C_6)$ consists of a length-6 anticycle. The *transitive reduction* $\vec{H}^-$ of $\vec{H}$ is a minimal subgraph of $\vec{H}$ whose transitive closure is $\vec{H}$.

Theorem 1. *A digraph $\vec{H}$ is the biclique comparability digraph of a bipartite graph if and only if $\vec{H}$ is asymmetric and transitive and every induced anticycle of length four is the rim of an oriented wheel isomorphic to the digraph on the right in Fig. 2.*

Proof. First suppose $\vec{H} = \vec{H}(G)$ where G is bipartite. We have already noted that $\vec{H}$ is asymmetric and transitive. Suppose C is an induced anticycle of length four in $\vec{H}(G)$. Say the vertices of C are

$$V_k = \frac{\{r_i : i \in I_k\}}{\{c_j : j \in J_k\}}, \quad \text{for } 1 \leqslant k \leqslant 4, \quad \text{where } I_1 \subset I_2 \supset I_3 \subset I_4 \supset I_1$$

(so $J_1 \supset J_2 \subset J_3 \supset J_4 \subset J_1$). Set $I = I_1 \cup I_3$ and $J = J_2 \cup J_4$. If $i \in I$ and $j \in J$, then $i \in I_2 \cap I_4$ and so r_i is adjacent to c_j in G (with the edge r_ic_j contributing to V_2 or V_4). Thus $\{r_i : i \in I\} \cup \{c_j : j \in J\}$ induces a bipartite subgraph of G and so there will be $I^+ \supseteq I$ and $J^+ \supseteq J$ such that $\{r_i : i \in I^+\} \cup \{c_j : j \in J^+\}$ induces a maximal complete bipartite subgraph of G. Then the two set containments

$$\{r_i : i \in I^+\} \supset \{r_i : i \in I_1\} \cup \{r_i : i \in I_3\},$$
$$\{c_j : j \in J^+\} \supset \{c_j : j \in J_2\} \cup \{c_j : j \in J_4\}$$

imply that the vertex $\frac{\{r_i : i \in I^+\}}{\{c_j : j \in J^+\}}$ of $\vec{H}(G)$ together with the vertices of C induce a subgraph isomorphic to the oriented wheel in Fig. 2.

Conversely, suppose $\vec{H}$ is any asymmetric transitive digraph. Assign sets of row vertices r_i to the vertices of $\vec{H}$ recursively as follows: To each *sink vertex* (vertex of out-degree zero) of $\vec{H}^-$ assign a distinct row vertex r_i and to each non-sink vertex assign the union of the sets previously assigned to its out-neighbors together with one new r_i. Similarly—except using *source vertices* (vertices of in-degree zero) and in-neighbors—assign sets of column vertices c_j to the vertices of $\vec{H}$. The r_is and c_js can be taken as the vertices a new bipartite graph G, with $r_i c_j \in E(G)$ if and only if r_i and c_j are in the sets assigned to a common vertex of $\vec{H}$. Since each vertex of $\vec{H}$ contributes an edge $r_i c_j$ to G that is assigned only to that vertex, distinct vertices of $\vec{H}$ correspond to distinct maximal complete bipartite subgraphs of G. Thus, $\vec{H}$ can be shown to be $\vec{H}(G)$ by showing that every maximal complete bipartite subgraph of G corresponds to a vertex of $\vec{H}$.

To show this (by contraposition), suppose $\vec{H} = \vec{H}(G)$ and G contains a maximal complete bipartite subgraph with row vertices $\{r_i : i \in I\}$ and column vertices $\{c_j : j \in J\}$, where $\frac{\{r_i : i \in I\}}{\{c_j : j \in J\}} \notin V(\vec{H})$ and $\min\{|I|, |J|\}$ is as small as possible. Since all the edges of each $K_{1,k}$ subgraph of G are in a common vertex of $\vec{H}$, both $|I|, |J| \geqslant 2$. Let I_1 [respectively, J_1] be a nonempty proper subset of I [and J]. The minimality of $\min\{|I|, |J|\}$ implies that there are four maximal complete bipartite subgraphs of G (four vertices of $V(\vec{H})$)

$$\frac{\{r_i : i \in I_1\}}{\{c_j : j \in J^+\}}, \frac{\{r_i : i \in I^+\}}{\{c_j : j \in J_1\}}, \frac{\{r_i : i \in (I - I_1)\}}{\{c_j : j \in J^\oplus\}}, \text{ and } \frac{\{r_i : i \in I^\oplus\}}{\{c_j : j \in (J - J_1)\}},$$

where $I \subseteq I^+ \cap I^\oplus$ and $J \subseteq J^+ \cap J^\oplus$ (the possibility of enlarging I and J is needed to get *maximal* complete bipartite subgraphs). Those four vertices induce an anticycle in $\vec{H}$, but this anticycle cannot be the rim of an oriented wheel isomorphic to the graph in Fig. 2, since $\frac{\{r_i : i \in I\}}{\{c_j : j \in J\}}$ is the only possible the center vertex for such a wheel. □

A bipartite graph is *chordal bipartite* [2,7] if every cycle of length at least six has a *chord* (meaning an edge joining nonconsecutive vertices); the bipartite graphs shown in Figs. 1 and 2 are examples. For a chordal bipartite G, [6] shows that $|V(\vec{H}(G))| \leqslant m = |E(G)|$ and that all the inclusion-maximal complete bipartite subgraphs of G can be listed in $\mathcal{O}(\min\{m \log n, n^2\})$ time. The remainder of Section 1 will study biclique comparability graphs of chordal bipartite graphs. Conjecture 6 in Section 2 will explain the special role that chordal bipartite graphs seem to play in the context of biclique comparability graphs and matrix analysis.

Theorem 2. *A bipartite graph G is chordal bipartite if and only if $H(G)$ contains no induced cycle of length greater than or equal to six* (*in other words, if and only if $\vec{H}(G)$ contains no induced anticycle of length greater than or equal to six*).

Proof. Suppose G is a bipartite graph that is not chordal bipartite—say G contains an induced cycle $C_G = (r_1, c_1, r_2, c_2, \ldots, r_k, c_k, r_1)$ of length $2k > 4$; set $r_{k+1} = r_1$ and $c_{k+1} = c_1$. For each $v \in V(G)$, let $N[v] = N(v) \cup \{v\}$. For each $i \in \{1, \ldots, k\}$, let B_i^r and B_i^c be the vertex sets of maximal complete bipartite subgraphs of G such that $N[r_i] \subseteq B_i^r$ and $N[c_i] \subseteq B_i^c$. So $c_i \in N[r_i] \subseteq B_i^r$ and $r_{i+1} \in N[c_i] \subseteq B_i^c$. Notice that $B_1^r, \ldots, B_k^r, B_1^c, \ldots, B_k^c$ are vertices of $H = H(G)$. If $r' \in B_i^r$, then $r' \in N(c_i) \subseteq B_i^c$, and so $B_i^r B_i^c \in E(H)$. If $c' \in B_i^c$, then $c' \in N(r_{i+1}) \subseteq B_{i+1}^r$, and so $B_i^c B_{i+1}^r \in E(H)$. Therefore $C_H = (B_1^r, B_1^c, B_2^r, B_2^c, \ldots, B_k^r, B_k^c, B_1^r)$ is a cycle of length $2k > 4$ in H. The cycle C_H cannot have a chord $B_i^r B_j^r$ with $i \neq j$, since that would require $r_i \in B_j^r$ or $r_j \in B_i^r$, making one of r_i and r_j be adjacent to all the neighbors of the other (contradicting the maximality of B_i^r or G_j^r). Similarly, C_H cannot have a chord $B_i^c B_j^c$ with $i \neq j$. Finally, C_H cannot have a chord $B_i^r B_j^c$ with $|i - j| \geqslant 1$, since that would require $r_i c_j \in E(G)$ (contradicting C_G being an induced cycle). Thus, C_H would be an induced cycle of length $\geqslant 6$ in H.

Conversely, suppose G is bipartite and $H = \vec{H}(G)$ contains an induced cycle $C_H = (B_1, B_2, \ldots, B_k, B_1)$ of even length $k \geqslant 6$; set $B_{k+1} = B_1$ and $B_0 = B_k$. Without loss of generality, we can assume that, when i is odd, $r_j \in B_i \Rightarrow r_j \in B_{i-1} \cap B_{i+1}$ and, when i is even, $c_j \in B_i \Rightarrow c_j \in B_{i-1} \cap B_{i+1}$. Since B_1 is not adjacent to B_4 in H, there must be some row vertex $r_1 \in B_1 - B_4$, and so, since r_1 is adjacent to every column vertex in B_1, r_1 must be nonadjacent to some $c_4 \in B_4 - B_1$. Similarly, some $c_2 \in B_2 - B_5$ must be nonadjacent to some $r_5 \in B_5 - B_2$, and some $r_3 \in B_3 - B_6$ must be nonadjacent to some $c_6 \in B_6 - B_3$. For each odd $i \in \{7, \ldots, k-1\}$ choose an $r_i \in B_i$ and for each even $i \in \{8, \ldots, k\}$ choose a $c_i \in B_i$. Then $C_G = (r_1, c_2, \ldots, r_{k-1}, c_k, r_1)$ is a cycle in G and there is an induced cycle C'_G of length at least six such that $\{r_1, c_2, r_3, c_4, r_5, c_6\} \subseteq V(C'_G) \subseteq V(C_G)$. Thus, G would not be chordal bipartite. □

$$M = \begin{bmatrix} A & C & ? & F \\ B & ? & ? & G \\ ? & D & E & H \end{bmatrix}$$

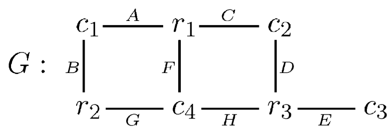

Fig. 3. A partial block matrix *M* and the corresponding bipartite graph *G*; each edge of *G* is labeled with the corresponding entry from *M*.

A graph is *weakly chordal* [2,7] if neither it nor its complement contains an induced cycle of length at least five. Therefore, a graph is chordal bipartite if and only if it is both bipartite and weakly chordal, and so *a bipartite graph is chordal bipartite if and only if it is weakly chordal*. The following is then a consequence of Theorem 2.

Corollary 3. *A bipartite graph G is chordal bipartite if and only if the biclique comparability graph $H(G)$ is weakly chordal.*

Proof. If *G* is chordal bipartite, then Theorem 2 shows that $H = H(G)$ contains no induced cycles of length $\geqslant 5$. Since *H* is automatically a comparability graph, its complement is a co-comparability graph and so contains no induced cycles of length $\geqslant 5$ [2, Theorem 7.2.7]. Thus *H* is weakly chordal. The converse follows directly from Theorem 2. □

For any digraph $\vec{H}$ on vertex set $\{h_1, \ldots, h_k\}$, the *bipartite transform* [4] is the undirected bipartite graph on vertex set $\{h'_1, \ldots, h'_k; h''_1, \ldots, h''_k\}$ whose only edges are $h'_i h''_j$ when there is an arc from h_i to h_j in $\vec{H}$; also see [1]. If $\vec{H}$ is transitively oriented, then [4] shows that *H* is weakly chordal if and only if its bipartite transform is chordal bipartite. Thus Corollary 3 also shows that a bipartite graph *G* is chordal bipartite if and only if the bipartite transform of $\vec{H}(G)$ is chordal bipartite.

2. Finding minimum ranks of partial matrices

Suppose *M* is a *partial matrix*, meaning a matrix in which certain entries are specified elements of a field and the other entries, denoted by ?s, are unspecified elements of the field. If, instead, the specified entries are blocks of specified elements and the unspecified entries are blocks of unspecified elements, then M is a *partial block matrix*. Fig. 3 (which is example (6) from [3]) shows a partial block matrix *M* in which $\begin{bmatrix} A & F \\ B & G \end{bmatrix}$ is an example of a *fully-specified submatrix*. Fig. 3 also shows the corresponding bipartite graph *G* whose adjacency matrix has 1s for the specified entries of *M* and 0s for the unspecified entries (r_i corresponds to the *i*th row and c_j to the *j*th column).

As another example, the chordal bipartite graph shown in Fig. 1 corresponds to the partial block matrix

$$\begin{bmatrix} A & D & H & ? & ? \\ B & E & ? & ? & ? \\ ? & F & I & ? & ? \\ C & G & J & M & P \\ ? & ? & K & N & Q \\ ? & ? & L & O & R \end{bmatrix}.$$

Correspondence 1. *The maximal fully specified submatrices of M correspond one-to-one with the inclusion-maximal complete bipartite subgraphs of G.*

If *G* is the bipartite graph corresponding to the partial block matrix *M*, then the digraph $\vec{H} = \vec{H}(G)$ can also be viewed as $\vec{H}(M)$ where the vertices are now the maximal fully specified submatrices of *M*, with an arc from vertex M_i to vertex M_j if and only if the row set of M_i (as rows of *M*) is (properly) contained in the row set of M_j; see Fig. 4.

Following [3], call a partial matrix M' a *subpattern* of a partial matrix *M* if M' is a partial submatrix of *M* in which some of the specified entries have been replaced by ?s; for example, the M' shown below is a subpattern

$$M' = \begin{bmatrix} A & C & F \\ B & ? & G \\ ? & ? & H \end{bmatrix}, \quad M'_{\triangleright} = \begin{bmatrix} F & A & C \\ G & B & ? \\ H & ? & ? \end{bmatrix}$$

of the *M* shown in Fig. 3. Call a partial matrix *M* (or the corresponding bipartite graph *G*) *triangular* if there is a permutation of *M* that produces a matrix $M_{\triangleright}$ such that, if the (i, j)-entry of $M_{\triangleright}$ is specified and if $i' \leqslant i$ and $j' \leqslant j$, then the (i', j')-entry of $M_{\triangleright}$ is specified. (Equivalently, if the (i, j)- and (p, q)-entries of *M* are specified, then at least one of the (i, q)- and (p, j)-entries of

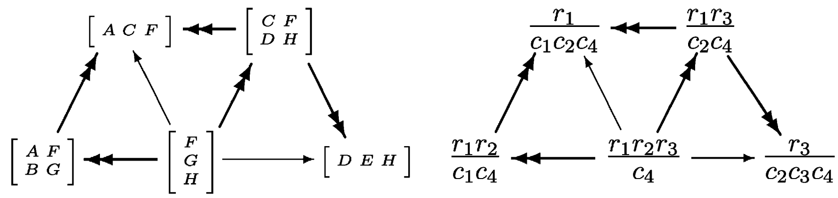

Fig. 4. Both the $\vec{H}(M)$ and $\vec{H}(G)$ versions of the biclique comparability digraph that corresponds to the M and G in Fig. 3; double-headed arcs lie along the maximal length paths and form the transitive reduction of $\vec{H}$.

$$\begin{bmatrix} F\\ G\\ H \end{bmatrix} \xrightarrow[\begin{bmatrix} F\\ G \end{bmatrix}]{} \begin{bmatrix} A & F\\ B & G \end{bmatrix} \xrightarrow[\begin{bmatrix} A & F \end{bmatrix}]{} \begin{bmatrix} A & C & F \end{bmatrix}$$

Fig. 5. One of the three maximal paths of $\vec{H}(M)$ from Fig. 4; each edge M_iM_{i+1} of the path is labeled with the matrix $M_i \cap M_{i+1}$.

M must also be specified.) For instance, the matrix M in Fig. 3 is not triangular, but its subpattern M' shown above is triangular (being permutable into the matrix $M'_{\triangleright}$ shown).

The *maximal paths* of a digraph are the source-to-sink directed paths in its transitive reduction; see Fig. 5.

Correspondence 2. *The maximal triangular subpatterns of M correspond one-to-one with the maximal paths of* $\vec{H}(M)$.

For instance, the maximal path P in Fig. 5 corresponds to the maximal triangular subpattern M' (given above) of M. Each edge M_iM_{i+1} along the path $P = \langle M_1, \ldots, M_k \rangle$ corresponds to the matrix $M_i \cap M_{i+1}$ in the simple sense illustrated by

$$\begin{bmatrix} A & F\\ B & G \end{bmatrix} \cap [\,A \quad C \quad F\,] = [\,A \quad F\,].$$

Similarly defining the union of partial matrices as illustrated by

$$\begin{bmatrix} A & F\\ B & G \end{bmatrix} \cup [\,A \quad C \quad F\,] = \begin{bmatrix} A & C & F\\ B & ? & G \end{bmatrix},$$

the path P corresponds to the triangular subpattern $M_1 \cup \cdots \cup M_k$ of M. Also, the maximality of the path P in $\vec{H}$ ensures that the corresponding subpattern is maximal triangular. The maximal path $\langle \frac{r_1r_2r_3}{c_4}, \frac{r_1r_2}{c_1c_4}, \frac{r_1}{c_1c_2c_4} \rangle$ of $\vec{H}(G)$ also corresponds to the maximal triangular subpattern M'.

Following [3], the *minimum rank* of a partial (block) matrix M is the minimum rank among all *completions* of M—all matrices of the same dimensions as M whose entries agree with the specified entries of M. When M is triangular, its minimum rank can be determined from the ranks of all the fully specified submatrices of M by a formula in [8]. Proposition 4 translates this formula into our terminology and notation.

Proposition 4 (*Woerdeman*). *If triangular matrix M corresponds to the maximal path* $P = \langle M_1, \ldots, M_k \rangle$ *of* $\vec{H}(M)$ *as in Correspondence* 2, *then the minimum rank of M is*

$$\sum_{M_i \in V(P)} \mathrm{rk}(M_i) \quad - \sum_{M_iM_j \in E(P)} \mathrm{rk}(M_i \cap M_j).$$

As an example, the subpattern M' that corresponds to the maximal path $\langle \frac{r_1r_2r_3}{c_4}, \frac{r_1r_2}{c_1c_4}, \frac{r_1}{c_1c_2c_4} \rangle$ in Fig. 4 (the path in Fig. 5) has minimum rank

$$\mathrm{rk}\begin{bmatrix} F\\ G\\ H \end{bmatrix} - \mathrm{rk}\begin{bmatrix} F\\ G \end{bmatrix} + \mathrm{rk}\begin{bmatrix} A & F\\ B & G \end{bmatrix} - \mathrm{rk}\,[A \;\; F] + \mathrm{rk}\,[A \;\; C \;\; F].$$

Again following [3], the *triangular minimum rank* of a partial (block) matrix M is the maximum, among all triangular subpatterns M' of M, of the minimum rank of M'. Every M has minimum rank at least as large as its triangular minimum rank.

Proposition 5 (*Cohen, Johnson, Rodman and Woerdeman*). *The triangular minimum rank of M is*

$$\max\left\{\sum_{M_i\in V(P)}\mathrm{rk}(M_i)-\sum_{M_iM_j\in E(P)}\mathrm{rk}(M_i\cap M_j)\colon P \text{ a maximal path in } \vec{H}(M)\right\}.$$

Cohen et al. [3] show that if the minimum rank and the triangular minimum rank of *M* are the same, then *G* is chordal bipartite. The following appears in [3] (also see [5]).

Conjecture 6 (*Cohen, Johnson, Rodman and Woerdeman*). *A partial block matrix M has minimum rank equal to*

$$\max\left\{\sum_{M_i\in V(P)}\mathrm{rk}(M_i)-\sum_{M_iM_j\in E(P)}\mathrm{rk}(M_i\cap M_j)\colon P \text{ a maximal path in } \vec{H}(M)\right\}.$$

if and only if the corresponding bipartite graph G is chordal bipartite.

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